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GENERALIZED LAPLACIANS AND MULTIPLE TRIGONOMETRIC SERIES

BY

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ABSTRACT. V. L. Shapiro gave a k-variable analogue for Riemann's theorem on formal integration of trigonometric series. This paper derives Shapiro's results with weaker conditions on the coefficients of the series and extends the results to series which are Bochner-Riesz summable of larger order.

1. **Introduction.** Let $f(x)$, $x = (x_1, \cdots, x_k)$, be a complex valued function defined in a neighborhood of $x_0 \in E^k$, $k \geq 2$. We will say $f(x)$ has at x_0 a generalized rth Laplacian equal to s if $f(x)$ is integrable over each sphere $|x - x_0| = t$, for t small, and if

$$(2\pi)^{-k/2} \int_{|y|=1} f(x_0 + ty)\,ds(y) = a_0 + a_2 t^2 + \cdots + a_{2r} t^{2r} + o(t^{2r})$$

as $t \to 0$, where $a_{2r} = s/2^{k/2+2r-1} r!\,\Gamma(r + k/2)$. This definition is due to Shapiro [4], who showed that if all partial derivatives of f of order $2r$ exist and are continuous in a neighborhood of x_0, then the generalized rth Laplacian of $f(x_0)$ exists and equals $\Delta^r f(x_0)$.

Let

$$T\colon \sum c_n e^{in\cdot x} \tag{1.1}$$

be a trigonometric series in k variables, with $n = (n_1, \cdots, n_k)$, $x = (x_1, \cdots, x_k)$, $n \cdot x = n_1 x_1 + \cdots + n_k x_k$, and $|n| = (n_1^2 + \cdots + n_k^2)^{1/2}$. For $\alpha \geq 0$ we denote by

$$\sigma_R^\alpha(x) = \sum_{|n|<R} \left(1 - \left(\frac{|n|}{R}\right)^2\right)^\alpha c_n e^{in\cdot x} \tag{1.2}$$

the (spherical) Bochner-Riesz means of order α of (1.1). We say T is summable (BR, α) at x to sum s if $\lim_{R\to\infty} \sigma_R^\alpha(x) = s$.

Shapiro [4] proved the following result, which is a k-dimensional analogue of Riemann's theorem on twice integrated trigonometric series (see, for example, [5, Vol. I, p. 319]).

Suppose $T\colon \Sigma c_n e^{in\cdot x}$ is (BR, m) summable at x_0 to the finite sum s,

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where m *is an integer* ≥ 0, *and where* $c_n = O(|n|^{\alpha})$, *for some* $\alpha < m + 2 - k$, *as* $|n| \to \infty$. *Let* r *be an integer* $\geq m/2 + 1$. *Set*

$$F(x) = \frac{c_0(x_1 + \cdots + x_k)^{2r}}{k^r[(2r)!]} + \sum_{|n| \neq 0} \frac{(-1)^r c_n}{|n|^{2r}} e^{in\cdot x}. \tag{1.3}$$

Then the generalized rth Laplacian of $F(x)$ *exists at* x_0 *and is equal to* s.

The purpose of this paper is to prove the conclusion of Shapiro's theorem under weaker hypotheses than those stated above. We are able to reduce the hypothesis on the decay of the coefficients of (1.1) and also on the order of summability required. Our result is

Theorem. *Let the series* (1.1) *be summable* (BR, β) *at* x_0 *to a finite sum* s, *where* β *is a real number* ≥ 0. *Let* r *be an integer such that* $r > \frac{1}{2}(\beta + 1)$, *and suppose* $\Sigma |n|^{\gamma} |c_n|^2 < \infty$ *for some* $\gamma > k - 1 - 4r$. *Then* $F(x)$ *defined by*

$$F(x) = \frac{c_0(x_1 + \cdots + x_k)^{2r}}{k^r[(2r)!]} + \lim_{R\to\infty} \sum_{0<|n|<R} (-1)^r \frac{c_n}{|n|^{2r}} e^{in\cdot x}$$

has a generalized rth Laplacian at x_0 *equal to sum* s.

We note, in particular, that our reduced condition on the growth of the coefficients c_n no longer implies that the series defining $F(x)$ converges uniformly. Hence our improvement extends Shapiro's result to the case when $F(x)$ is not necessarily continuous.

2. Before beginning the proof of the Theorem, we note that the Theorem is true if T is only a constant term; see, for example, [2, p. 289]. We may therefore assume $c_0 = 0$. We will also assume, as we may, that $x_0 = 0$, $s = 0$, and that r is the smallest integer with $2r > \beta + 1$. Write $\beta = m + \alpha$ where m is an integer and $0 \leq \alpha < 1$.

Write $S_R = S_R(0) = \Sigma_{|n|<R} c_n$ and for $\beta > 0$ put

$$S_R^{\beta} = \frac{1}{\Gamma(\beta)} \int_0^R (R - u)^{\beta - 1} S_u \, du. \tag{2.1}$$

S_R^{β}, as a function of R, is the fractional integral of order β of $f(R) = S_R$. Hardy has shown that $\sigma_R^{\beta}(x) \to s$ if and only if $\bar{\sigma}_R^{\beta}(x) \to s$, where

$$\bar{\sigma}_R^{\beta}(x) = \frac{\beta}{R^{\beta}} \int_0^R S_u(x)(R - u)^{\beta - 1} \, du.$$

Thus the series (1.1) is (BR, β) summable to zero at $x = 0$ if and only if

$$S_R^{\beta} = o(R^{\beta}), \tag{2.2}$$

as $R \to \infty$.

3. We begin with some lemmas. Lemmas 1 and 2 are modifications of lemmas from [4].

Lemma 1. *Assume* $\Sigma|n|^{\gamma}|c_n|^2 < \infty$ *for some* $\gamma > k - 4r - 1$ *and suppose* Σc_n *is* $(BR, m+1)$ *summable to zero. Then* $S_R^n = o(R^{2r+\frac{1}{2}})$ *for* $n = 0, \cdots, m$, *as* $R \longrightarrow \infty$.

Proof. Write $\gamma = k - 4r - 1 + \epsilon$, where $\epsilon > 0$. Then

$$\sum_{|n|<R} |c_n| = \sum_{|n|<R} |n|^{\gamma/2}|c_n|\,|n|^{-\gamma/2} \leq \left(\sum |n|^{\gamma}|c_n|^2\right)^{\frac{1}{2}}\left(\sum_{|n|<R} |n|^{-\gamma}\right)^{\frac{1}{2}} \tag{3.1}$$

$$= C(R^{-\gamma+k})^{\frac{1}{2}} = CR^{\frac{1}{2}(-k+4r+1-\epsilon+k)} = o(R^{2r+\frac{1}{2}}).$$

Note that for $j = 1, 2, \cdots, m+2$,

$$\sum_{|s|<R} c_s(R - |s| + j)^{m+1} = \sum_{|s|<R+j} - \sum_{R \leq |s| < R+j} = \text{I} + \text{II}.$$

$$\text{I} = o(R^{m+1}) \quad \text{since } \sum c_s \text{ is } (BR, m+1) \text{ summable to } 0.$$

$$|\text{II}| \leq C \sum_{|s|<R+j} |c_s| = o(R^{2r+\frac{1}{2}}), \quad \text{by (3.1)}.$$

Since $2r > \beta + 1 \geq m + 1$, we have, combining I and II, $\Sigma_{|s|<R} c_s(R - |s| + j)^{m+1} = o(R^{2r+\frac{1}{2}})$.

We observe that there exist numbers A_{jn} for $j = 1, \cdots, m+2$ and $n = 0, \cdots, m$ such that $\Sigma_{j=1}^{m+2} A_{jn}(z + j)^{m+1} = z^n$ for $z \in \mathbf{C}$. This is Lemma 3 of [4]. Hence

$$S_R^n = \frac{1}{n!} \sum_{|s|<R} c_s(R - |s|^n) = \frac{1}{n!} \sum_{|s|<R} c_s \sum_{j=1}^{m+2} A_{jn}(R - |s| + j)^{m+1}$$

$$= \sum_{j=1}^{m+2} \frac{1}{n!} A_{jn} \sum_{|s|<R} c_s(R - |s| + j)^{m+1} = \sum_{j=1}^{m+2} \frac{1}{n!} A_{jn} o(R^{2r+\frac{1}{2}}) = o(R^{2r+\frac{1}{2}}).$$

Lemma 2. *Let* $J_s(t)$ *denote the Bessel function of the first kind of order* s, *where* s *is an integer or half integer. Then for each* $n > 0$,

$$\frac{d^n}{dt^n} \frac{J_s(t)}{t^{s+2r}} = O(t^{-s-2r-\frac{1}{2}}),$$

as t *tends to infinity.*

Proof. The argument is the same as the proof of Lemma 2 of [4]. However we use here the fact that $J_s(t) = O(t^{-\frac{1}{2}})$ as $t \longrightarrow \infty$.

Lemma 3. *If* $0 \leq \alpha < 1$, *then* $\int_0^u (u - z)^{-\alpha} z^{\alpha} dz = O(u)$ *as* u *tends to infinity.*

Proof.

$$\int_0^u (u-z)^{-\alpha} z^\alpha\, dz = \int_0^{u/2} + \int_{u/2}^u = A + B,$$

where

$$A = \int_0^{u/2} (u-z)^{-\alpha} z^\alpha\, dz = O(u^{-\alpha}) \int_0^{u/2} z^\alpha dz = O(u),$$

$$B = \int_{u/2}^u (u-z)^{-\alpha} z^\alpha\, dz = O(u^\alpha) \int_{u/2}^u (u-z)^{-\alpha}\, dz = O(u^\alpha) O(u^{1-\alpha}) = O(u).$$

Lemma 4. *With* S_u^β *defined by* (2.1), *then for almost all* u

$$S_u^m = \frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz.$$

Proof. Put

$$I = \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz = \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{(1-\alpha)-1} S_z^{\alpha+m}\, dz.$$

Thus I is the fractional integral of order $(1-\alpha)$ of $S_u^{\alpha+m}$. But since $S_u^{\alpha+m}$ is the fractional integral of order $(\alpha+m)$ of S_u, therefore I is the fractional integral of S_u of order $(\alpha+m)+(1-\alpha) = m+1$. Hence for almost all u, $dI/du = S_u^m$.

4. **Proof of the Theorem.** We must show

$$(2\pi)^{-k/2} \int_{z\in\Sigma} F(tz)\, ds(z) = a_0 + a_2 t^2 + \cdots + a_{2(r-1)} t^{2(r-1)} + o(t^{2r})$$

as t tends to zero.

The hypothesis on the moduli of c_n implies $\Sigma |n|^{k-1+\epsilon} |c_n |n|^{-2r}|^2 < \infty$, for some $\epsilon > 0$. Therefore by virtue of Theorem 1 of [3] the series defining $F(x)$ in (1.3) converges spherically almost everywhere on each sphere $|x| = t$. By virtue of Theorem 2 of [3], with $p = 1$, we may integrate this series term by term over each sphere.

Bochner [1] observed $(2\pi)^{-k/2} \int_{z\in\Sigma} e^{int\cdot z} ds(z) = J_s(|n|t)/(|n|t)^s$, where $s = ½(k-2)$. Hence,

$$(2\pi)^{-k/2} \int_\Sigma F(tz)\, ds(z) = \lim_{R\to\infty} \sum_{0<|n|<R} (-1)^r \frac{c_n}{|n|^{2r}} (2\pi)^{-k/2} \int_\Sigma e^{in\cdot tz} ds(z)$$

(4.1)

$$= \lim_{R\to\infty} (-1)^r \sum_{0<|n|<R} \frac{c_n}{|n|^{2r}} \frac{J_s(|n|t)}{(|n|t)^s} = (-1)^r t^{2r} \lim_{R\to\infty} \sum_{0<|n|<R} c_n\, \gamma(|n|t),$$

where $\gamma(u) = u^{-s-2r} J_s(u)$.

We apply summation by parts to $\Sigma_{|n|<R}\, c_n \gamma(|n|t)$ m times.

$$\sum_{|n|<R} c_n\gamma(|n|t) = S_R\gamma(Rt) - \int_0^R S_u \frac{d}{du}\gamma(ut)\,du$$

$$= S_R\gamma(Rt) - S_R^1 \frac{d}{dR}\gamma(Rt) + \int_0^R S_u^1 \frac{d^2}{du^2}\gamma(ut)\,du$$

$$\vdots$$

$$= S_R\gamma(Rt) - S_R^1\frac{d}{dR}\gamma(Rt) + \cdots + (-1)^m S_R^m \frac{d^m}{dR^m}\gamma(Rt)$$

$$+ (-1)^{m+1}\int_0^R S_u^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du.$$

Using Lemmas 1 and 2, with $j = 1, \cdots, m$, $S_R^j(d^j/dR^j)\gamma(Rt) = o(R^{2r+\frac{1}{2}}) \cdot O(R^{-(k-2)/2-2r-\frac{1}{2}}) = o(1)$, as R tends to infinity. Thus,

$$\lim_{R\to\infty} \sum_{0<|n|<R} c_n\gamma(|n|t) = (-1)^{m+1}\int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du,$$

and returning to (4.1) we get

$$(2\pi)^{-k/2}\int_\Sigma F(tz)\,ds(z) = (-1)^r t^{2r} \sum_{0<|n|<R} c_n\gamma(|n|t) \tag{4.2}$$

$$= (-1)^{r+m+1}t^{2r}\int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\gamma(ut)\,du.$$

We use the formula $J_s(z) = \Sigma_{n=0}^\infty((-1)^n(\frac{1}{2}z)^{s+2n}/n!\,\Gamma(s+n+1))$. Hence

$$z^{-s}J_s(z) = \sum_{n=0}^\infty \frac{(-1)^n(2)^{-s-2n}z^{2n}}{n!\Gamma(s+n+1)}$$

$$= c_0 + c_2z^2 + \cdots + c_{2n}z^{2n} + \cdots.$$

Define $P(z) = c_0 + c_2z^2 + \cdots + c_{2(r-1)}z^{2(r-1)}$ and let $\lambda(z) = (z^{-s}J_s(z) - P(z))/z^{2r}$. Then $\lambda(z)$ is an entire function in the plane and

$$\gamma(z) = \lambda(z) + z^{-2r}P(z). \tag{4.3}$$

Substituting into (4.2),

$$(2\pi)^{-k/2}\int_\Sigma F(tz)\,ds(z) = (-1)^{r+m+1}t^{2r}\int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\left[\frac{P(ut)}{(ut)^{2r}} + \lambda(ut)\right]du$$

$$= (-1)^{r+m+1}\int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\frac{P(ut)}{u^{2r}}\,du + (-1)^{r+m+1}t^{2r}\int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}\lambda(ut)\,du \tag{4.4}$$

$$= A(t) + t^{2r}B(t).$$

$$A(t) = (-1)^{r+m+1} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \frac{P(ut)}{u^{2r}}\, du$$

$$(4.5) \qquad = (-1)^{r+m+1} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \left\{ \sum_{j=0}^{r-1} c_j \frac{(ut)^{2j}}{u^{2r}} \right\} du$$

$$= (-1)^{r+m+1} \sum_{j=0}^{r-1} t^{2j} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} c_j u^{2(j-r)}\, du = \sum_{j=0}^{r-1} A_{2j} t^{2j}.$$

We will show A_{2j} is finite for each j.

$$A_{2j} = c_j' \int_0^\infty S_u^m u^{2(j-r)-m-1} du = c_j' \int_1^\infty S_u^m u^{-n_j}\, du,$$

with $m + 3 \leq n_j \leq 2r + m + 1$. By Lemma 4,

$$S_u^m = (1/\Gamma(1-\alpha))(d/du) \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz,$$

for almost all u. Hence

$$A_{2j} = c_j'' \int_1^\infty u^{-n_j} \frac{d}{du} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz\, du.$$

Integrating by parts,

$$A_{2j} = c_j'' u^{-n_j} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz \Big|_1^\infty - c_j'' \int_1^\infty u^{-n_j-1} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz\, du = \alpha_j + \beta_j.$$

By (2.2), we have $S_R^{\alpha+m} = o(R^{\alpha+m})$, so

$$\alpha_j = c_j'' u^{-n_j} \int_0^u (u-z)^{-\alpha} o(z^{\alpha+m})\, dz \Big|_1^\infty$$

$$= o(u^{-n_j+m}) \int_0^u (u-z)^{-\alpha} z^\alpha\, dz \Big|_1^\infty = o(u^{-n_j+m}) O(u) \Big|_1^\infty$$

by Lemma 3. Since $n_j \geq m+3$, $\alpha_j = O(1)$.

To estimate β_j we again use Lemma 3.

$$\beta_j = c_j'' \int_1^\infty u^{-n_j-1} \int_0^u (u-z)^{-\alpha} o(z^{\alpha+m})\, dz\, du$$

$$= c_j'' \int_1^\infty u^{-n_j-1} o(u^m) \int_0^u (u-z)^{-\alpha} o(z^\alpha)\, dz\, du$$

$$= c_j'' \int_1^\infty o(u^{-n_j-1+m}) O(u)\, du = O(1).$$

This shows that each $A_{2j} = \alpha_j + \beta_j$ is finite. Combining this last statement with (4.4) and (4.5), we obtain

$$(2\pi)^{-k/2} \int_\Sigma F(tz)\, ds(z) = \sum_{j=0}^{r-1} A_{2j} t^{2j} + t^{2r} B(t).$$

5. The proof of the Theorem will be complete when we show $B(t)$ tends to zero with t. Using Lemma 4,

$$B(t) = (-1)^{r+m+1} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du$$

$$= (-1)^{r+m+1} \int_0^\infty \frac{1}{\Gamma(\alpha+m)} \frac{d}{du} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz \frac{d^{m+1}}{du^{m+1}} \lambda(ut)\, du.$$

(We are justified in replacing the limits of integration in the outer integral by $0 \leq u < \infty$ because $S_u^m = 0$ for $0 \leq u < 1$.) We integrate the expression for $B(t)$ by parts.

$$B(t) = \frac{(-1)^{r+m+1}}{\Gamma(\alpha+m)} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz \cdot \frac{d^{m+1}}{du^{m+1}} \lambda(ut) \Big|_0^\infty$$

$$+ \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz \cdot \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du$$

$$= B_1(t) + B_2(t).$$

We assert that, for $t \neq 0$, $B_1(t)$ is zero. Since $\lambda(z)$ is entire in the plane, $d^{m+1}\lambda(ut)/du^{m+1}$ remains bounded as u tends to zero. Hence at zero the expression defining $B_1(t)$ is zero. To evaluate the expression defining $B_1(t)$ at $u = \infty$ we use (4.3) and (2.2).

$$\int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz \cdot \frac{d^{m+1}}{du^{m+1}} \lambda(ut) = o(u^m) \int_0^u (u-z)^{-\alpha} z^\alpha\, dz \cdot \frac{d^{m+1}}{du^{m+1}} \left[\gamma(ut) - \frac{P(ut)}{(ut)^{2r}} \right].$$

The integral in this last expression is $O(u)$ by Lemma 3. By Lemma 2, $d^{m+1}\gamma(ut)/du^{m+1} = O(u^{-m-1})$, and since $P(ut)(ut)^{-2r}$ is a quotient of polynomials with the degree of the denominator greater by two than the degree of the numerator, $(d^{m+1}/du^{m+1})\, P(ut)/(ut)^{2r} = O(u^{-2-m-1})$ as $u \to \infty$. Hence, near $u = \infty$, the expression defining $B_1(t)$ is $o(u^m)O(u)[O(u^{-m-1}) + O(u^{-m-1})] = o(1)$.

It remains to be shown that $B_2(t)$ tends to zero with t.

$$B_2(t) = \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m}\, dz \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du$$

$$= \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty S_z^{\alpha+m} \left\{ \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du \right\} dz$$

$$= \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty S_z^{\alpha+m} H(z,t)\, dz = \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \left\{ \int_0^{1/t} + \int_{1/t}^\infty \right\} = \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \{M + N\}.$$

We first consider M. In this integral z ranges between 0 and $1/t$, and

$$H(z,t) = \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)\, du = \int_z^{1/t} + \int_{1/t}^\infty = P + Q.$$

$\lambda(z)$ is entire, so for $|ut| < 1$, $d^{m+2}\lambda(ut)/du^{m+2} = O(t^{m+2})$. Hence

$$(5.2)\qquad \begin{aligned} P &= \int_z^{1/t} (u-z)^{-\alpha} O(t^{m+2}) du = O(t^{m+2})(1/t - z)^{1-\alpha} \\ &= O(t^{m+1})(1/t-z)^{1-\alpha}(t(1/t-z)) = O(t^{m+1})(1/t-z)^{1-\alpha}, \end{aligned}$$

since $t(1/t - z) = 1 - tz \leq 1$.

$$\begin{aligned} Q &= \int_{1/t}^{\infty} (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\ &= \int_{1/t}^{\infty} (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du + \int_{1/t}^{\infty} (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \frac{P(ut)}{(ut)^{2r}} du = Q_1 + Q_2. \end{aligned}$$

By Lemma 2,

$$\frac{d^{m+2}}{du^{m+2}} \gamma(ut) = t^{m+2}\, O((ut)^{-s-2r-1/2}) = t^{m+3/2-2r-s} O(u^{-s-2r-1/2}).$$

Hence,

$$\begin{aligned} Q_1 &= \int_{1/t}^{\infty} (u-z)^{-\alpha} t^{m+3/2-2r-s} O(u^{-s-2r-1/2}) du \\ &= (1/t-z)^{-\alpha} t^{m+3/2-2r-s} \int_{1/t}^{\infty} O(u^{-s-2r-1/2}) du = (1/t-z)^{-\alpha} O(t^{m+1}). \end{aligned}$$

Since $(d^{m+2}/du^{m+2})[P(ut)(ut)^{-2r}] = t^{-2}O(u^{-m-4})$, therefore

$$Q_2 = \int_{1/t}^{\infty} (u-z)^{-\alpha} t^{-2} O(u^{-m-4}) du = (1/t-z)^{-\alpha} O(t^{-2+m+3}).$$

Combining the estimates for Q_1 and Q_2 we get $Q = (1/t - z)^{-\alpha} O(t^{m+1})$, and with (5.1) and (5.2) we conclude, for $0 \leq z \leq 1/t$, $H(z, t) = (1/t - z)^{-\alpha} O(t^{m+1})$. Therefore

$$\begin{aligned} M &= \int_0^{1/t} S_z^{\alpha+m} H(z,t) dz = \int_0^{1/t} o(z^{\alpha+m}) O(t^{m+1})(1/t-z)^{-\alpha} dz \\ &= o(1/t)^{\alpha+m} t^{m+1} \int_0^{1/t} (1/t - z)^{-\alpha} dz = o(1). \end{aligned}$$

We now consider $N = \int_{1/t}^{\infty} S_z^{\alpha+m} H(z, t) dz$. In this integral z is larger than $1/t$.

$$\begin{aligned} H(z,t) &= \int_z^{\infty} (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\ &= \int_z^{\infty} (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du + \int_z^{\infty} (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \frac{P(ut)}{(ut)^{2r}} du = R + S. \end{aligned}$$

To compute R we apply Lemma 2.

$$(d^{m+2}/du^{m+2})\gamma(ut) = t^{m+3/2-s-2r} O(u^{-s-2r-1/2}).$$

Hence,

$$R = t^{m+3/2-s-2r}\int_z^\infty (u-z)^{-\alpha} O(u^{-s-2r-\frac{1}{2}})\,du$$

$$= t^{m+3/2-s-2r}\left[\int_z^{2z} + \int_{2z}^\infty\right].$$

$$\int_z^{2z} (u-z)^{-\alpha} O(u^{-s-2r-\frac{1}{2}})\,du = O(z^{-s-2r-\frac{1}{2}})\int_z^{2z} (u-z)^{-\alpha}\,du$$

$$= O(z^{-s-2r-\frac{1}{2}}) O(z^{1-\alpha}) = O(z^{-s-2r+\frac{1}{2}-\alpha}),$$

$$\int_{2z}^\infty (u-z)^{-\alpha} O(u^{-s-2r-\frac{1}{2}})\,du = O(z^{-\alpha})\int_{2z}^\infty u^{-s-2r-\frac{1}{2}}\,du$$

$$= O(z^{-\alpha}) O(z^{-s-2r+\frac{1}{2}}) = O(z^{-s-2r+\frac{1}{2}-\alpha}).$$

Thus, $R = t^{m+3/2-s-2r} O(z^{-s-2r+\frac{1}{2}-\alpha}) = t^{-\frac{1}{2}} O(z^{-\alpha-m-3/2})(tz)^{m+2-s-2r}$. Since $|tz| \geq 1$ in the interval of integration and since $2r \geq m+2$ and $s \geq 0$, the last factor on the right of this last equation is ≤ 1. Hence,

$$R = t^{-\frac{1}{2}} O(z^{-\alpha-m-3/2}). \tag{5.3}$$

To compute S, we again use the fact that $(d^{m+2}/du^{m+2})P(ut)/(ut)^{2r} = t^{-2}O(u^{-m-4})$; and therefore

$$S = \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \frac{P(ut)}{(ut)^{2r}}\,du$$

$$= t^{-2}\int_z^\infty (u-z)^{-\alpha} O(u^{-m-4})\,du = t^{-2}\left\{\int_z^{2z} + \int_{2z}^\infty\right\}.$$

$$\int_z^{2z} (u-z)^{-\alpha} O(u^{-m-4})\,du = O(z^{-m-4})\int_z^{2z} (u-z)^{-\alpha}\,du$$

$$= O(z^{-m-4}) O(z^{1-\alpha}) = O(z^{-m-\alpha-3}).$$

$$\int_{2z}^\infty (u-z)^{-\alpha} O(u^{-m-4})\,du = O(z^{-\alpha})\int_{2z}^\infty u^{-m-4}\,du = O(z^{-\alpha}) z^{-m-3}.$$

Thus $S = t^{-2} O(z^{-m-\alpha-3}) = t^{-\frac{1}{2}} O(z^{-m-\alpha-3/2})$. Combining with (5.3), we see that for $1/t < z$, $H(z, t) = R + S = t^{-\frac{1}{2}} O(z^{-m-\alpha-3/2})$.

Substituting into the integral defining N,

$$N = \int_{1/t}^\infty S_z^{\alpha+m} H(z, t)\,dz = \int_{1/t}^\infty o(z^{\alpha+m}) t^{-\frac{1}{2}} O(z^{-m-\alpha-3/2})\,dz$$

$$= t^{-\frac{1}{2}}\int_{1/t}^\infty o(z^{-3/2})\,dz = t^{-\frac{1}{2}} o(1/t)^{-\frac{1}{2}} = o(1).$$

We have shown that $B_2(t)$ tends to zero with t. This completes the proof of the Theorem.

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